

On the Anti-Wishart distribution

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We provide the probability distribution function of matrix elements each of which is the inner product of two vectors. The vectors we are considering here are independently distributed but not necessarily Gaussian variables. When the number of components M of each vector is greater than the number of vectors N , one has a $N \times N$ symmetric matrix. When $M \geq N$ and the components of each vector are independent Gaussian variables, the distribution function of the $N(N+1)/2$ matrix elements was obtained by Wishart in 1928. When $N > M$, what we called the “Anti-Wishart” case, the matrix elements are no longer completely independent because the true degrees of freedom becomes smaller than the number of matrix elements. Due to this singular nature, analytical derivation of the probability distribution function is much more involved than the corresponding Wishart case. For a class of general random vectors, we obtain the analytical distribution function in a closed form, which is a product of various factors and delta function constraints, composed of various determinants. The distribution function of the matrix element for the $M \geq N$ case with the same class of random vectors is also obtained as a by-product. Our result is closely related to and should be valuable for the study of random magnet problem and information redundancy problem.

I. INTRODUCTION

Many problems in physics [1] can be related to the matrix problem that we will discuss. The matrix we shall consider in this work takes a special form: each matrix element, $Y_{i,j}$ is the inner product of two independent random vectors. Historically speaking, this type of random matrix came into research literature even before the now-called random matrix theory first introduced and explored by Wigner, Dyson, Mehta and others [2]. The number of degrees of freedom is important for this class of matrices. Denote N the dimension of square matrix \mathbf{Y} , M the dimension of the vectors. Depending which is larger, the resultant matrices can be singular with lots of zero eigenvalues; or normally behaved.

The original motivation to study this type of distribution function mainly come from the effort to understand the correlations of fluctuations. But in recent years, interests have manifested in studying the case where matrix elements are not independent, but inner products of vector-pairs. To be more precise, let us consider the following precise definition. Mathematically, we may denote N vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N$, each of which lives in M dimensions. For example, vector \vec{x}_i has its components $x_i^1, x_i^2, \dots, x_i^M$. After shifting by the average in each sample, and possible rescaling, we may assume that the vector components x_i^α (for all $i \in \{1, 2, \dots, N\}$, $\alpha \in \{1, 2, \dots, M\}$) are random variables with some distribution function $P(\{x_i^\alpha\})$. We then define the Matrix of interest \mathbf{Y} whose matrix element Y_{ij} is simply defined as $\vec{x}_i \cdot \vec{x}_j$.

$$Y_{ij} \equiv \vec{x}_i \cdot \vec{x}_j = \sum_{\alpha=1}^M x_i^\alpha x_j^\alpha. \quad (1)$$

When $M > N$ and the background distribution $P(\{x_i^\alpha\})$ being a Gaussian

$$P(\{x_i^\alpha\}) = \left[\frac{\det^{1/2}[\sigma]}{(2\pi)^{N/2}} \right]^M \exp\left(-\frac{1}{2} \sum_{i,j=1}^N \vec{x}_i \cdot \vec{x}_j \sigma_{i,j}\right), \quad (2)$$

Wishart obtained [3] in 1928 a compact expression for the distribution function of $P(\{Y_{ij}\})$. The result looks simplest when the matrix $[\sigma]$ is an identity matrix, and we have

$$P(\{Y_{ij}\}) = \mathcal{N}^{-1} \det[\mathbf{Y}]^{(M-N-1)/2} \exp(-\text{Tr } \mathbf{Y}/2) \quad (3)$$

where \mathcal{N} is some normalization constant. For the case of general $[\sigma]$, we replace the $\exp(-\text{Tr } \mathbf{Y}/2)$ part by $\exp(-\sum_{i,j=1}^N \sigma_{i,j} Y_{i,j}/2)$. A general re-derivation can be found in [4]. This result has been the fundamentals (and one of the triumphs) of multivariate statistical inference. When the components of the vectors are not Gaussian variables, this is still a challenging problem not to mention the generic case in the opposite direction $M < N$, which we coined as “Anti-Wishart” case.

In this paper, we will extend the analytical result to a class of more general $P(\{x_i^\alpha\})$ and also to the “Anti-Wishart” case. Basically, we will consider the case where

$$P(\{x_i^\alpha\}) = \prod_{i=1}^N f_i(\vec{x}_i \cdot \vec{x}_i). \quad (4)$$

Note that each vector \vec{x}_i is allowed to have a different spherical distribution function. For example, we allow f_1 being a Gaussian, f_2 being a delta function $\mathcal{C}\delta(\vec{x}_2 \cdot \vec{x}_2 - 1)$ constraining the vector \vec{x}_2 to have unit length etc.

In Wishart’s case, the matrix elements Y_{ij} of \mathbf{Y} have enough degrees of freedom. All the eigenvalues are nonzero generically. This is so because the original degrees of freedom, MN is larger than the number of Y_{ij} s, N^2 . The case for $M < N$, however, eluded our reach for many years. Partly due to the difficulty in dealing with singular measures and partly due the lack of motivations. However, recent advances in many branches of science have necessitated quantitative knowledge about distribution function of this sort. One simple example come from the study of bio-molecular interaction matrix, e.g., the protein-protein interaction matrix that is now intensively studied in molecular biology. The knowledge of such matrix is extremely important to quantitatively understand how cell function, etc. Another example come from the scenario in global knowledge network proposed by Maslov and Zhang [5] where information redundancy is exploited. Finally, in the random magnetic system the coupling J_{ij} between two spins \mathbf{S}_i and \mathbf{S}_j could be random variables obtained form inner product of two vectors \vec{x}_i and \vec{x}_j that characterize the property at each specific sites i and j . This actually happens while transforming a

two dimensional XY random field magnetic model into a random bond one [6]. In this case, it will be very desirable to have knowledge of such distribution function.

In this paper, for the class of random vector distribution (4), we document down *for the first time* the *correct* exact distribution of $\{Y_{ij}\}$ in analytical form using only fundamental tools of linear algebra. To have the best flow in showing the derivation, we will state two useful lemmas and introduce useful notations in the next section followed by another section devoted to the derivation. The proofs of the two lemma are relegated to the appendix.

II. TWO LEMMAS

Before we get to the derivation of the distribution function, we would like to introduce some useful notation and state two useful lemmas. First, let us denote $\Delta_{i_1, i_2, \dots, i_K}$ as the determinant of a compactified matrix obtained by eliminating from \mathbf{Y} matrix elements whose both indices are not completely in the set $\{i_1, i_2, \dots, i_K\}$. For convenience, we define $\Delta_{1,2,\dots,L-1} = 1$ and $\Delta_{1,2,\dots,L-1,j} = Y_{jj}$ when $L = 1$. Naturally the following abbreviations $\Delta_{1,2,\dots,L-1} = \Delta_0$ and $\Delta_{1,2,\dots,L-1,j} = \Delta_{0,j} = \Delta_j$ when $L = 1$ apply.

The notation $\langle i + j \rangle$ denotes a single index with the following rules: $Y_{i\langle j+k \rangle} \equiv Y_{ij} + Y_{ik}$ and $Y_{\langle j+k \rangle \langle j+k \rangle} = Y_{jj} + Y_{kj} + Y_{jk} + Y_{kk}$. Having introduced such notations, we now state the two useful lemmas.

Lemma 1 Provided the matrix \mathbf{Y} is symmetric, using the above definitions we have

$$\begin{aligned} & 4 \Delta_{1,2,\dots,L-1,k} \cdot \Delta_{1,2,\dots,L-1,j} - \\ & - \left(\Delta_{1,2,\dots,L-1,\langle k+j \rangle} - \Delta_{1,2,\dots,L-1,k} - \Delta_{1,2,\dots,L-1,j} \right)^2 \\ & = 4 \Delta_{1,2,\dots,L-1} \cdot \Delta_{1,2,\dots,L-1,k,j} \end{aligned} \tag{5}$$

Lemma 2 Using the above definition and assume the matrix \mathbf{Y} being symmetric, we have

$$\Delta_{1,2,\dots,L-1,j} \cdot \left[\Delta_{1,2,\dots,L-1,\langle k+l \rangle} - \Delta_{1,2,\dots,L-1,k} - \Delta_{1,2,\dots,L-1,l} \right]$$

$$\begin{aligned}
& -\frac{1}{2} \left[\Delta_{1,2,\dots,L-1,\langle j+k \rangle} - \Delta_{1,2,\dots,L-1,k} - \Delta_{1,2,\dots,L-1,j} \right] \cdot \\
& \quad \cdot \left[\Delta_{1,2,\dots,L-1,\langle j+l \rangle} - \Delta_{1,2,\dots,L-1,k} - \Delta_{1,2,\dots,L-1,l} \right] \\
& = \Delta_{1,2,\dots,L-1} \cdot \\
& \quad \cdot \left[\Delta_{1,2,\dots,L-1,j,\langle k+l \rangle} - \Delta_{1,2,\dots,L-1,j,k} - \Delta_{1,2,\dots,L-1,j,l} \right]
\end{aligned} \tag{6}$$

III. DERIVATION

In this section, under the Gaussian background distribution we will derive the Anti-Wishart distribution and also obtain the Wishart distribution as a by-product. Formally, we write the distribution function as

$$P(\{Y_{ij}\}) = \int d\vec{x}_1 d\vec{x}_2 \cdots d\vec{x}_N \left[\prod_{i=1}^N f_i(\vec{x}_i \cdot \vec{x}_i) \right] \prod_{i \leq j} \delta(Y_{ij} - \vec{x}_i \cdot \vec{x}_j) \tag{7}$$

where $\vec{x}_i \cdot \vec{x}_j = \sum_{\alpha=1}^M x_i^\alpha x_j^\alpha$ is the inner product of vector i and vector j . When the components of each vector are independent Gaussian random variables, one has

$$\prod_{i=1}^N f_i(\vec{x}_i \cdot \vec{x}_i) = (2\pi)^{-MN/2} \exp \left[-\frac{1}{2} \sum_{i=1}^N \sum_{\alpha=1}^M (x_i^\alpha)^2 \right]. \tag{8}$$

We will come back to it when compare to the Wishart result.

Because we want to go beyond the case where components of each vector are independent Gaussian random variables, our strategy is to integrate one solid angle at a time, see below. Since the matrix elements are invariant under rotation in the M dimensional space, we can choose $\vec{x}_1 \parallel \hat{e}_M$ and write the components of the rest of other $N-1$ vectors in polar angles, i.e., write

$$\begin{aligned}
x_i^M &= r_i \cos \theta_{M-1;i} \\
x_i^{M-1} &= r_i \sin \theta_{M-1;i} \cos \theta_{M-2;i} \\
x_i^{M-2} &= r_i \sin \theta_{M-1;i} \sin \theta_{M-2;i} \cos \theta_{M-3;i} \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
x_i^2 &= r_i \sin \theta_{M-1;i} \sin \theta_{M-2;i} \cdots \sin \theta_{2;i} \cos \theta_{1;i} \\
x_i^1 &= r_i \sin \theta_{M-1;i} \sin \theta_{M-2;i} \cdots \sin \theta_{2;i} \sin \theta_{1;i}
\end{aligned} \tag{9}$$

where the notation $\theta_{a;i}$ represents the a th polar angle of the i th vector. Under such decomposition, the volume element

$$d\vec{x}_i \rightarrow r_i^{M-1} dr_i d\Omega_i(M) = r_i^{M-1} dr_i \prod_{a=1}^{M-1} [\sin \theta_{a;i}^{a-1} d\theta_{a;i}] \tag{10}$$

where $0 \leq \theta_{2 \leq a \leq M-1; 2 \leq i \leq N} < \pi$ and $0 \leq \theta_{1; 2 \leq i \leq N} < 2\pi$ are the polar angles and are bounded by the expression above. Naturally, the solid angle element $d\Omega_i(b)$ in b dimension is given by

$$d\Omega_i(b) \equiv \prod_{a=1}^{b-1} [\sin \theta_{a;i}^{a-1} d\theta_{a;i}]. \tag{11}$$

One can immediately integrate away first the N radial vectors to get rid of the delta functions contain only diagonal elements of the matrix. We then integrate away the solid angles $\theta_{M-1; 2 \leq i \leq N}$. After the first stage, we have

$$\begin{aligned}
P(\{Y_{ij}\}) &= 2^{-N} \left[\prod_{i=1}^N f_i(Y_{ii}) \right] \int \prod_{i=1}^N [Y_{ii}^{\frac{M-2}{2}} d\Omega_i(M)] \prod_{j=2}^N \delta(Y_{1j} - \sqrt{Y_{11}Y_{jj}} \cos \theta_{M-1;j}) \\
&\cdot \prod_{2 \leq j < k \leq N} \delta(Y_{jk} - \sqrt{Y_{jj}Y_{kk}} I_{j,k}(M-1))
\end{aligned} \tag{12}$$

where

$$I_{j,k}(a) = \cos \theta_{a;j} \cos \theta_{a;k} + \sin \theta_{a;j} \sin \theta_{a;k} I_{j,k}(a-1) \tag{13}$$

with

$$I_{j,k}(a=0) = 1 \quad \text{for all } j, k. \tag{14}$$

Note that the factor $\prod_{i=1}^N f_i(Y_{ii})$ becomes $(2\pi)^{-MN/2} e^{-\frac{1}{2} \text{Tr} \mathbf{Y}}$ when the components of each vector are independent Gaussian variables. Note that in eq.(12) the delta functions do not depend on the polar angles of vector \vec{x}_1 , therefore we can integrate $d\Omega_1$ and obtain a factor K_M , area of unit sphere in M dimension. Now let us note that we may rewrite

$Y_{ij} = \frac{1}{2}[\Delta_{\langle i+j \rangle} - \Delta_i - \Delta_j] = \frac{1}{2}[\Delta_{0,\langle i+j \rangle} - \Delta_{0,i} - \Delta_{0,j}]$. We can therefore rewrite the expression (12) into

$$\begin{aligned}
P(\{Y_{ij}\}) &= 2^{-N} \left[\prod_{i=1}^N f_i(Y_{ii}) \right] K_M \left[\prod_{i=1}^N \Delta_i^{\frac{M-2}{2}} \right] \int \left[\prod_{i=2}^N d\Omega_i(M) \right] \\
&\quad \cdot \prod_{j=2}^N \delta \left(\frac{1}{2}[\Delta_{0,\langle 1+j \rangle} - \Delta_{0,1} - \Delta_{0,j}] - \sqrt{\Delta_{0,1}\Delta_{0,j}} \cos \theta_{M-1;j} \right) \\
&\quad \cdot \prod_{2 \leq j < k \leq N} \delta \left(\frac{1}{2}[\Delta_{0,\langle j+k \rangle} - \Delta_{0,j} - \Delta_{0,k}] - \sqrt{\Delta_{0,j}\Delta_{0,k}} I_{j,k}(M-1) \right) \quad (15)
\end{aligned}$$

We may then integrate away $\theta_{M-1;2 \leq j \leq N}$ using

$$\int_0^\pi \sin \theta_j d\theta_j = \frac{1}{\sqrt{\Delta_{0,1}\Delta_{0,j}}} \int_{-\sqrt{\Delta_{0,1}\Delta_{0,j}}}^{\sqrt{\Delta_{0,1}\Delta_{0,j}}} dy \quad (16)$$

where $y \equiv \sqrt{\Delta_{0,1}\Delta_{0,j}} \cos \theta_j$. Note that because of the delta functions

$$\begin{aligned}
\sqrt{\Delta_{0,1}\Delta_{0,j}} \cos \theta_{M-1;j} &= \frac{1}{2}[\Delta_{0,\langle 1+j \rangle} - \Delta_{0,1} - \Delta_{0,j}], \\
\sin \theta_{M-1;j} &= \sqrt{1 - \cos^2 \theta_{M-1;j}} = \sqrt{\frac{\Delta_0 \Delta_{0,1,j}}{\Delta_{0,1}\Delta_{0,j}}} \quad (17)
\end{aligned}$$

where lemma 1 and $0 \leq \theta_{M-1;j} < \pi$ are used in the sine part of the above equation.

Therefore the integral over polar angles $\{\theta_{M-1;j}\}_{j=2}^N$ yields

$$\begin{aligned}
P(\{Y_{ij}\}) &= 2^{-N} \left[\prod_{i=1}^N f_i(Y_{ii}) \right] K_M \Delta_1^{\frac{(N-2)(N-M+1)}{2}} \left[\prod_{j=2}^N \Delta_{1,j}^{\frac{M-3}{2}} \right] \int \left[\prod_{i=2}^N d\Omega_i(M-1) \right] \\
&\quad \prod_{2 \leq j < k \leq N} \delta \left(\frac{1}{2}[\Delta_{1,\langle j+k \rangle} - \Delta_{1,j} - \Delta_{1,k}] - \sqrt{\Delta_{1,j}\Delta_{1,k}} I_{j,k}(M-2) \right) \quad (18)
\end{aligned}$$

where lemma 2 and $\delta(x/a) = a \delta(x)$ are used.

A moment of reflection tells us that $I_{j,k}(M-2)$ is nothing but setting all the $\theta_{M-1;j(k)} = \pi/2$ so that the M th component of vectors \vec{x}_j and \vec{x}_k are identically zero. Or equivalently, we are then looking at vectors living in $M-1$ dimensional space instead of M dimensional space. Since the solid angle of vector \vec{x}_1 and all the radial components of the vectors are completely integrated out, we are now left with $N-1$ *unit vectors* living in $M-1$ dimensions. We can then again require that the $M-1$ th component of vector \vec{x}_2 to be along \hat{e}_{M-1} so that $x_2^{M-1} = 1$ and $x_2^{1 \leq \alpha \leq M-2} = 0$. We then again write the components

of the other unit vectors in polar angles such as in (9) but with $\theta_{M-1; 2 \leq i \leq N} = \pi/2$ and $\theta_{M-2; 2} = 0$. Note that spherical symmetry guarantees that $d\Omega_i(M-1)$ has exactly the same form regardless how one chooses the axes. Under such consideration, we note again that $I_{2, 3 \leq k \leq N} = \cos \theta_{M-2; k}$ and the delta functions again give us that

$$\begin{aligned} \sqrt{\Delta_{1,2}\Delta_{1,k}} \cos \theta_{M-2; k} &= \frac{1}{2}[\Delta_{1, \langle 2+k \rangle} - \Delta_{1,2} - \Delta_{1,k}] \\ \sin \theta_{M-2; k} &= \sqrt{1 - \cos^2 \theta_{M-2; k}} = \sqrt{\frac{\Delta_1 \Delta_{1,2,k}}{\Delta_{1,2}\Delta_{1,k}}}, \end{aligned} \quad (19)$$

again we have used $0 \leq \theta_{M-2; k} < \pi$. After such understanding, we may then proceed to integrate the polar angles $\theta_{M-2; 3 \leq i \leq N}$ and, noting that the solid angle integration of $d\Omega_2(M-1)$ leads to K_{M-1} , obtain

$$\begin{aligned} P(\{Y_{ij}\}) &= 2^{-N} \left[\prod_{i=1}^N f_i(Y_{ii}) \right] K_M K_{M-1} \Delta_{1,2}^{\frac{(N-3)(N-M+1)}{2}} \left[\prod_{j=3}^N \Delta_{1,2,j}^{\frac{M-4}{2}} \right] \int \left[\prod_{i=3}^N d\Omega_i(M-2) \right] \\ &\quad \prod_{3 \leq j < k \leq N} \delta \left(\frac{1}{2}[\Delta_{1,2, \langle j+k \rangle} - \Delta_{1,2,j} - \Delta_{1,2,k}] - \sqrt{\Delta_{1,2,j}\Delta_{1,2,k}} I_{j,k}(M-3) \right) \\ &= 2^{-N} \left[\prod_{i=1}^N f_i(Y_{ii}) \right] K_M K_{M-1} \cdots K_{M-L+2} \Delta_{1,2,\dots,L-1}^{\frac{(N-L)(N-M+1)}{2}} \left[\prod_{j=L}^N \Delta_{1,2,\dots,L-1,j}^{\frac{M-L-1}{2}} \right] \\ &\quad \int \left[\prod_{i=L}^N d\Omega_i(M-L+1) \right] \prod_{L \leq j < k \leq N} \delta \left(\frac{1}{2}[\Delta_{1,2,\dots,L-1, \langle j+k \rangle} - \Delta_{1,2,\dots,L-1,j} - \Delta_{1,2,\dots,L-1,k}] - \right. \\ &\quad \left. - \sqrt{\Delta_{1,2,\dots,L-1,j}\Delta_{1,2,\dots,L-1,k}} I_{j,k}(M-L) \right) \end{aligned} \quad (20)$$

Now we see how this process can continue with application of lemmas 1 and 2. When $N \leq M$, the process actually terminate at $L = N$ where all the delta functions have been integrated out. In this way, we have extended the celebrated result of Wishart to the more generic case

$$\begin{aligned} P(\{Y_{ij}\}) &= 2^{-N} \left[\prod_{i=1}^N f_i(Y_{ii}) \right] \left[\prod_{i=1}^N K_{M-i+1} \right] \Delta_{1,2,\dots,N}^{\frac{M-N-1}{2}} \\ &= \left[\prod_{i=1}^N \frac{f_i(Y_{ii}) K_{M-i+1}}{2} \right] [\det(\mathbf{Y})]^{(M-N-1)/2} \end{aligned} \quad (21)$$

and in the more restricted case with $\prod_{i=1}^N f_i(Y_{ii}) = (2\pi)^{-MN/2} \exp(-\frac{1}{2}\text{Tr}\mathbf{Y})$, we have exactly the Wishart result

$$P(\{Y_{ij}\}) = (2\pi)^{-MN/2} \left[\prod_{i=1}^N \frac{K_{M-i+1}}{2} \right] [\det(\mathbf{Y})]^{(M-N-1)/2} \exp(-\frac{1}{2} \text{Tr} \mathbf{Y}). \quad (22)$$

For the case of our interest $N > M$, however, the integral does not terminate that way and there will be leftover delta functions. The furthest we can go then is to integrate till $L = M$ together with one last complication that the range of angle $\theta_{1,i}$ is between 0 and 2π (instead of between 0 and π) and therefore $\sin \theta_{1,i}$ can take both positive and negative signs. To see it explicitly, we may integrate up to $L = M - 1$ and notice that $I_{j,k}(1) = \cos \theta_{1,j} \cos \theta_{1,k} + \sin \theta_{1,j} \sin \theta_{1,k}$ because of $I_{j,k}(0) = 1$. This way, we have

$$P(\{Y_{ij}\}) = 2^{-N} \left[\prod_{i=1}^N f_i(Y_{ii}) \right] K_M K_{M-1} \cdots K_3 \Delta_{1,2,\dots,M-2}^{\frac{(N-M+1)(N-M+1)}{2}} \int \left[\prod_{i=M-1}^N d\Omega_i(2) \right] \\ \prod_{M-1 \leq j < k \leq N} \delta \left(\frac{1}{2} [\Delta_{1,2,\dots,M-2,\langle j+k \rangle} - \Delta_{1,2,\dots,M-2,j} - \Delta_{1,2,\dots,M-2,k}] - \right. \\ \left. - \sqrt{\Delta_{1,2,\dots,M-2,j} \Delta_{1,2,\dots,M-2,k}} I_{j,k}(1) \right) \quad (23)$$

We then again choose the effectively the direction of the $(M-1)$ th unit vector to be along the \hat{e}_2 direction and therefore the solid angle of the new unit vector \vec{x}_{M-1} that lives in two dimensions. This way, we have

$$P(\{Y_{ij}\}) = 2^{-N} \left[\prod_{i=1}^N f_i(Y_{ii}) \right] \left[\prod_{j=2}^M K_j \right] \Delta_{1,2,\dots,M-1}^{\frac{(N-M)(N-M+1)}{2}} \left[\prod_{j=M}^N (\Delta_{1,2,\dots,M-1,j})^{-1/2} \right] \\ \sum_{\text{sgns}} \prod_{M \leq j < k \leq N} \delta \left(\frac{1}{2} [\Delta_{1,2,\dots,M-1,\langle j+k \rangle} - \Delta_{1,2,\dots,M-1,j} - \Delta_{1,2,\dots,M-1,k}] - \right. \\ \left. - \text{sgns} \sqrt{\Delta_{1,2,\dots,M-1,j} \Delta_{1,2,\dots,M-1,k}} \right) \quad (24)$$

where

$$\int_0^{2\pi} d\theta = \int_0^\pi -\frac{d \cos \theta}{\sin \theta} + \int_\pi^{2\pi} -\frac{d \cos \theta}{\sin \theta} = \frac{1}{|\sin \theta|} \left[\int_{-1}^1 d \cos \theta \mid_{\sin \theta \geq 0} + \int_{-1}^1 d \cos \theta \mid_{\sin \theta \leq 0} \right] \quad (25)$$

is used. Now the sum over “sgns” deserves some explanations. Each candidate of $\sin \theta_{1,i}$ in eq.(23) in principle can take both positive and negative values. This means that for each k , the quantity $\sqrt{\Delta_{1,2,\dots,M-1,k}}$ can carry both positive and negative signs. Because each of the remaining reduced unit vectors $\vec{x}_M, \vec{x}_{M+1}, \dots, \vec{x}_N$ can play a role in $\sqrt{\Delta_{1,2,\dots,M-1,k}}$, we

therefore have to consider 2^{N-M+1} different combinations. That is to say, our sum over signs actually consists of 2^{N-M+1} terms each of which is a product of $(N-M)(N-M+1)/2$ delta functions. In order to better organize these delta functions, we introduce two new notations:

$$b_k \equiv \sqrt{\Delta_{1,2,\dots,M-1,k}} \quad (26)$$

$$B_{k,l} \equiv \frac{1}{2}[b_{\langle k+l \rangle}^2 - b_k^2 - b_l^2]. \quad (27)$$

Under this new notation, we may rewrite our distribution function as

$$P(\{Y_{ij}\}) = 2^{-N} \left[\prod_{i=1}^N f_i(Y_{ii}) \right] \left[\prod_{j=2}^M K_j \right] \Delta_{1,2,\dots,M-1}^{\frac{(N-M)(N-M+1)}{2}} \left[\prod_{j=M}^N (\Delta_{1,2,\dots,M-1,j})^{-1/2} \right] \sum_{\{s_i=\pm 1\}} \prod_{M \leq j < k \leq N} \delta(B_{jk} - s_j s_k b_j b_k), \quad (28)$$

where $s_i = \pm 1$ are Ising variables conveniently introduced to represent the signs needed. Note that although eq.(28) could be regarded as the end result of integrations, to render it useful we will reassemble these 2^{N-M+1} combinations into a single term. This is in some way similar to obtain the partition function of an Ising system by summing up all possible spin configurations. We may also say that obtaining eq.(28) is only half way to our goal.

To work towards the final goal, we now start the task of reassembling these 2^{N-M+1} terms of product of delta functions. When applied to symmetric matrices, lemma 1 (with $L \rightarrow M$) tells us that

$$b_k^2 b_l^2 - B_{k,l}^2 = \Delta_{1,2,\dots,M-1} \cdot \Delta_{1,2,\dots,M-1,k,l}. \quad (29)$$

With the labeling of $k \in \{M, M+1, \dots, N\}$, we can now order the \pm signs carried by each b_k in the following manner. First, we observe that if we change the signs of every b_k , the delta function is invariant. This immediately leads to a two fold symmetry which allows us to require that the sign carried by b_N being always positive. Although there are in total 2^{N-M+1} terms in the sum, the two-fold symmetry dictates only 2^{N-M} different terms. These terms are selected by our choosing b_N always carrying positive sign, i.e.

$s_N = 1$. For the 2^{N-M} different terms, we generate them in the follows: We first write down the two cases where b_M can be either positive or negative. We organize it as

$$\begin{array}{c|c} & s_M \\ \hline & + \\ & - \end{array} \quad (30)$$

We then make two identical such copies. For the first copy we have b_{M+1} carrying positive signs, and for the second copy we have b_{M+1} carrying negative signs.

$$\begin{array}{c|cc} & s_{M+1} & s_M \\ \hline & + & + \\ & + & - \\ & - & + \\ & - & - \end{array} \quad (31)$$

We then make two identical copies of the above. We again in the first copy put in b_{M+2} that carries positive signs and in the second copy put in b_{M+2} that carries negative signs. We then arrives at

$$\begin{array}{c|ccc} & s_{M+2} & s_{M+1} & s_M \\ \hline & + & + & + \\ & + & + & - \\ & + & - & + \\ & + & - & - \\ & - & + & + \\ & - & + & - \\ & - & - & + \\ & - & - & - \end{array} \quad (32)$$

This process keeps going till we adding positive-sign-carrying b_{N-1} to first copy and negative-sign-carrying b_{N-1} to the second copy. After that we add positive-sign-carrying b_N to every term. We finally have something look like

	s_N	s_{N-1}	\cdots	s_{M+2}	s_{M+1}	s_M
1	+	+	\cdots	+	+	+
2	+	+	\cdots	+	+	-
3	+	+	\cdots	+	-	+
4	+	+	\cdots	+	-	-
5	+	+	\cdots	-	+	+
6	+	+	\cdots	-	+	-
7	+	+	\cdots	-	-	+
8	+	+	\cdots	-	-	-
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots
$2^{N-M} - 3$	+	-	\cdots	-	+	+
$2^{N-M} - 2$	+	-	\cdots	-	+	-
$2^{N-M} - 1$	+	-	\cdots	-	-	+
2^{N-M}	+	-	\cdots	-	-	-

(33)

We now start by combining terms $2l - 1$ and $2l$ for all $l \leq 2^{N-M-1}$. In order to simplify the notation, we shall only write out the $\sum_{\{s_i=\pm 1\}} \prod_{M \leq j < k \leq N} \delta(B_{jk} - s_j s_k b_j b_k)$ part and multiply the final results by appropriate factor later. In table (33), the first term has all the b_k carrying positive signs, i.e. $s_k = 1$, while the second term has b_M carrying negative sign but with the rest of b_k carrying positive signs. The sum of the first and the second term therefore look like

$$\left[\delta(B_{M,M+1} - b_M b_{M+1}) \prod_{k=M+2}^N \delta(B_{M,k} - b_M b_k) + \delta(B_{M,M+1} + b_M b_{M+1}) \prod_{k=M+2}^N \delta(B_{M,k} + b_M b_k) \right] \prod_{(M+1) \leq k < l \leq N} \delta(B_{k,l} - b_k b_l) \quad (34)$$

In the $\prod_{k=M+2}^N \delta(B_{M,k} - b_M b_k)$ part of the first term, we replace b_M by $B_{M,M+1}/b_{M+1}$ and b_k by $B_{M+1,k}/b_{M+1}$. For the $\prod_{k=M+2}^N \delta(B_{M,k} + b_M b_k)$ part of second term, we replace b_M by $-B_{M,M+1}/b_{M+1}$ and b_k by $B_{M+1,k}/b_{M+1}$. This way, the sum of the term 1 and term 2 can be rewritten as

$$\begin{aligned}
& [\delta(B_{M,M+1} - b_M b_{M+1}) + \delta(B_{M,M+1} + b_M b_{M+1})] \\
& \prod_{k=M+2}^N \delta(B_{M,k} - \frac{B_{M,M+1} B_{M+1,k}}{b_{M+1}^2}) \prod_{(M+1) \leq k < l \leq N} \delta(B_{k,l} - b_k b_l) \\
& = 2 \frac{b_M b_{M+1}}{\Delta_{1,2,\dots,M-1}} \delta(\Delta_{1,2,\dots,M-1,M,M+1}) \\
& \prod_{k=M+2}^N \delta(B_{M,k} - \frac{B_{M,M+1} B_{M+1,k}}{b_{M+1}^2}) \prod_{(M+1) \leq k < l \leq N} \delta(B_{k,l} - b_k b_l) \tag{35}
\end{aligned}$$

Similarly, the third and the fourth terms can also be added to

$$\begin{aligned}
& \left[\delta(B_{M,M+1} + b_M b_{M+1}) \prod_{k=M+2}^N \delta(B_{M,k} - b_M b_k) \delta(B_{M+1,k} - b_{M+1} b_k) \right. \\
& \left. + \delta(B_{M,M+1} - b_M b_{M+1}) \prod_{k=M+2}^N \delta(B_{M,k} + b_M b_k) \delta(B_{M+1,k} + b_{M+1} b_k) \right] \\
& \prod_{(M+1) \leq k < l \leq N} \delta(B_{k,l} - b_k b_l), \tag{36}
\end{aligned}$$

and with similar reasoning this sum leads to

$$\begin{aligned}
& 2 \frac{b_M b_{M+1}}{\Delta_{1,2,\dots,M-1}} \delta(\Delta_{1,2,\dots,M-1,M,M+1}) \prod_{k=M+2}^N \delta(B_{M,k} - \frac{B_{M,M+1} B_{M+1,k}}{b_{M+1}^2}) \\
& \prod_{k=M+2}^N \delta(B_{M+1,k} + b_{M+1} b_k) \prod_{(M+2) \leq k < l \leq N} \delta(B_{k,l} - b_k b_l). \tag{37}
\end{aligned}$$

One thing we notice immediately is that if we factor out the common factor between the sum of term 1 and 2 as well as the sum of term 3 and 4, we see an expression that is very similar to the original one as if the variable b_M does not come into the picture in the first place.

In fact the common factor

$$2 \frac{b_M b_{M+1}}{\Delta_{1,2,\dots,M-1}} \delta(\Delta_{1,2,\dots,M-1,M,M+1}) \prod_{k=M+2}^N \delta(B_{M,k} - \frac{B_{M,M+1} B_{M+1,k}}{b_{M+1}^2}) \tag{38}$$

is the same for every pairwise sum of $k = 2l - 1$ and $k = 2l$. The reason is very simple. Originally we have for $M + 2 \leq k \leq N$

$$\delta(B_{M,k} - s_M s_k b_M b_k) \tag{39}$$

in the product and we also have delta functions

$$\begin{aligned} & \delta(B_{M,M+1} - s_M s_{M+1} b_M b_{M+1}) \\ & \delta(B_{M+1,k} - s_{M+1} s_k b_{M+1} b_k) \end{aligned} \quad (40)$$

which give us

$$s_M b_M = \frac{B_{M,M+1}}{s_{M+1} b_{M+1}} \quad \text{and} \quad s_k b_k = \frac{B_{M+1,k}}{s_k b_k} \quad (41)$$

Upon substitution into eq.(39) we have

$$\delta\left(B_{M,k} - \frac{B_{M,M+1} B_{M+1,k}}{(s_{M+1})^2 b_{M+1}^2}\right) \quad (42)$$

and using the fact that $(s_k)^2 = 1$ for all $1 \leq k \leq N$. We thus proved that every pairwise sum produce such common factor. Furthermore, when we sum up those terms together, we reduced the number of terms exactly by half and at the same time gain a factor 2.

Remember that the sign of b_N is fixed to be positive, i.e. $s_N = +1$ and this process ends when turning $\delta(B_{N-1,N} - s_{N-1} b_{N-1} b_N) + \delta(B_{N-1,N} + s_{N-1} b_{N-1} b_N)$ into a single delta function, which corresponds to $k = N - 1$ and $l = N$. Note that we start with $k = M$, therefore, we have to do the same process for $N - M$ times. Together with the two-fold symmetry we mentioned at the very beginning, we will end up with

$$2 \cdot 2^{N-M} \left[\prod_{j=M}^{N-1} \frac{b_j b_{j+1}}{\Delta_{1,2,\dots,M-1}} \delta(\Delta_{1,2,\dots,M-1,j,j+1}) \right] \prod_{j=M}^{N-2} \prod_{k=j+2}^N \delta\left(B_{j,k} - \frac{B_{j,j+1} B_{j+1,k}}{b_{j+1}^2}\right) \quad (43)$$

To further simplify the expression, we see from lemma 2 that

$$B_{j,l} B_{l,k} - b_l^2 B_{j,k} = -\frac{1}{2} (\Delta_{1,2,\dots,M-1,l,\langle j+k \rangle} - \Delta_{1,2,\dots,M-1,l,j} - \Delta_{1,2,\dots,M-1,l,k}) \cdot \Delta_{1,2,\dots,M-1} \quad (44)$$

and with $l \rightarrow j + 1$, we see that

$$\begin{aligned} & \delta\left(B_{j,k} - \frac{B_{j,j+1} B_{j+1,k}}{b_{j+1}^2}\right) \\ & = \frac{\Delta_{1,2,\dots,M-1,j+1}}{\Delta_{1,2,\dots,M-1}} \delta\left(\frac{1}{2} [\Delta_{1,2,\dots,M-1,j+1,\langle j+k \rangle} - \Delta_{1,2,\dots,M-1,j+1,j} - \Delta_{1,2,\dots,M-1,j+1,k}]\right) \end{aligned} \quad (45)$$

We can now include the part we did not include explicitly and obtain finally

$$P(\{Y_{ij}\}) = \left[\prod_{i=1}^N f_i(Y_i) \right] \left[\prod_{j=1}^M \frac{K_j}{2} \right] \prod_{j=M+1}^{N-1} \Delta_{1,2,\dots,M-1,j}^{N-j-\frac{1}{2}} \prod_{j=M}^{N-1} \delta(\Delta_{1,2,\dots,M-1,j,j+1}) \prod_{j=M}^{N-2} \prod_{k=j+2}^N \delta\left(\frac{1}{2}[\Delta_{1,2,\dots,M-1,j+1,\langle j+k \rangle} - \Delta_{1,2,\dots,M-1,j+1,j} - \Delta_{1,2,\dots,M-1,j+1,k}]\right). \quad (46)$$

The reason that this final expression does not look very symmetric come from the fact that when we exploited the gauge degrees of freedom we chose \vec{x}_1 to be parallel to \hat{e}_M , \vec{x}_2 parallel to \hat{e}_{M-1} etc. Therefore, any permutation of the vectors in the order of integration should give identical results. When one consider that version, the symmetry will appear explicitly. The current expression, however, could be more useful from the standpoint of numerical use as will be explained in a separate publication [7] where more details will be presented. An alternative way to get the equivalent expression is to diagonalize the \mathbf{Y} matrix first, and then focus on the nonzero eigenvalues of the \mathbf{Y} matrix. Being seemingly more elegant, this way gives the same results and does not necessarily provide an easier way to calculate conditional probabilities for predicting redundant information. The application of our results as well as comparison of various approaches will be discussed in a later publication [7]. Because our method only assumes spherical symmetry in the ensemble of vectors, it can be applied to magnetic systems where the spins are of fixed lengths. Our method also bears potential to model the real knowledge network where the components of a vector might not be completely independent random Gaussian variables.

Finally, let us end with a brief note. After the calculational part of this work was completed, Janik and Nowak [8] recently presented a similar result which followed closely the route in [4]. Their method is simple but crucially depends on the random Gaussian ensemble assumed. Interestingly, the two results display a crucial discrepancy in terms of the number of constraints (delta functions). Although the degrees of freedom counting is somehow elementary, we would like to go over it again here. We first note that there are NM integration variables. However, the gauge symmetry assures that only $NM - [M(M-1)/2]$ of those integrations are independent with respect to the \mathbf{Y} matrix. Since the \mathbf{Y} matrix has $N(N+1)/2$ independent matrix elements, we therefore find that the

final number of delta functions should be $N(N+1)/2 - \{NM - [M(M-1)/2]\} = (N-M)(N-M+1)/2$. This expected number of delta functions indeed show up naturally in our final expression.

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APPENDIX

Let us first prove lemma 1.

For illustration purpose, we draw the matrix determinant represented by $\Delta_{1,2,\dots,L-1,\langle k+j \rangle}$ as

$$\det \begin{pmatrix} Y_{11} & Y_{12} & \cdots & Y_{1,L-1} & Y_{1k} + Y_{1j} \\ Y_{21} & Y_{22} & \cdots & Y_{2,L-1} & Y_{2k} + Y_{2j} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Y_{L-1,1} & Y_{L-1,2} & \cdots & Y_{L-1,L-1} & Y_{L-1,k} + Y_{L-1,j} \\ Y_{k1} + Y_{j1} & Y_{k2} + Y_{j2} & \cdots & Y_{k,L-1} + Y_{j,L-1} & Y_{kk} + Y_{jj} + Y_{kj} + Y_{jk} \end{pmatrix}. \quad (47)$$

From above illustration, we have

$$\begin{aligned} \Delta_{1,2,\dots,L-1,\langle k+j \rangle} &= [Y_{kk} + Y_{jj} + Y_{jk} + Y_{kj}] \Delta_{1,2,\dots,L-1} \\ &+ \sum_{n=1}^{L-1} (-1)^{L+n} [Y_{kn} + Y_{jn}] \tilde{A}_{L,n}(\langle k+j \rangle) \end{aligned} \quad (48)$$

where $\tilde{A}_{L,n}(\langle k+j \rangle)$ denotes the minor of the matrix element on row L column n in the matrix shown above. Note that the elements on the L th column are given by $Y_{n \in \{1,2,\dots,L-1\}, \langle k+j \rangle}$ and the elements in the L th row are given by $Y_{\langle k+j \rangle, n \in \{1,2,\dots,L-1\}}$. Similarly,

$$\begin{aligned}\Delta_{1,2,\dots,L-1,k} &= \sum_{n=1}^{L-1} Y_{kn} (-1)^{L+n} \tilde{A}_{L,n}(k) \\ &\quad + Y_{kk} \Delta_{1,2,\dots,L-1}\end{aligned}\tag{49}$$

and an identical expression exist for $\Delta_{1,2,\dots,L-1,j}$. One immediate observation is that

$$\tilde{A}_{L,n}(\langle k+j \rangle) = \tilde{A}_{L,n}(k) + \tilde{A}_{L,n}(j).\tag{50}$$

This part can be easily seen if we calculate each minor by expanding along its last column $L-1$: summing elements on row $L-1$ and column $1 \leq n \leq L-1$ multiply by their corresponding subminors. Note that

$$\tilde{A}_{L,n}(\langle k+j \rangle) = \det \begin{pmatrix} Y_{11} & Y_{12} & \cdots & Y_{1,n-1} & Y_{1,n+1} & \cdots & Y_{1k} + Y_{1j} \\ Y_{21} & Y_{22} & \cdots & Y_{2,n-1} & Y_{2,n+1} & \cdots & Y_{2k} + Y_{2j} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ Y_{L-1,1} & Y_{L-1,2} & \cdots & Y_{L-1,n-1} & Y_{L-1,n+1} & \cdots & Y_{L-1,k} + Y_{L-1,j} \end{pmatrix},\tag{51}$$

and

$$\tilde{A}_{L,n}(k) = \det \begin{pmatrix} Y_{11} & Y_{12} & \cdots & Y_{1,n-1} & Y_{1,n+1} & \cdots & Y_{1k} \\ Y_{21} & Y_{22} & \cdots & Y_{2,n-1} & Y_{2,n+1} & \cdots & Y_{2k} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ Y_{L-1,1} & Y_{L-1,2} & \cdots & Y_{L-1,n-1} & Y_{L-1,n+1} & \cdots & Y_{L-1,k} \end{pmatrix}.\tag{52}$$

Since the subminors associated with such decomposition are exactly the same whether the elements at the last column is $\langle k+j \rangle$ or just k or l .

Now let us re-express the quantity of interest

$$\begin{aligned}\Delta_{1,2,\dots,L-1,\langle k+j \rangle} &- \Delta_{1,2,\dots,L-1,k} - \Delta_{1,2,\dots,L-1,j} \\ &= \sum_{n=1}^{L-1} (-1)^{L+n} \left[Y_{kn} (\tilde{A}_{L,n}(\langle k+j \rangle)) \right. \\ &\quad \left. - \tilde{A}_{L,n}(k) + Y_{jn} (\tilde{A}_{L,n}(\langle k+j \rangle) - \tilde{A}_{L,n}(j)) \right] \\ &\quad + [Y_{kj} + Y_{jk}] \cdot \Delta_{1,2,\dots,L-1} \\ &= \sum_{n=1}^{L-1} (-1)^{L+n} \left[Y_{kn} \tilde{A}_{L,n}(j) + Y_{jn} \tilde{A}_{L,n}(k) \right] \\ &\quad + [Y_{kj} + Y_{jk}] \cdot \Delta_{1,2,\dots,L-1}\end{aligned}\tag{53}$$

And consequently, we have the square of the above expression as

$$\begin{aligned}
& [\Delta_{1,2,\dots,L-1,\langle k+j \rangle} - \Delta_{1,2,\dots,L-1,k} - \Delta_{1,2,\dots,L-1,j}]^2 \\
&= [Y_{jk} + Y_{kj}]^2 \cdot \Delta_{1,2,\dots,L-1}^2 + 2 [Y_{jk} + Y_{kj}] \cdot \Delta_{1,2,\dots,L-1} \sum_{n=1}^{L-1} (-1)^{L+n} [Y_{kn} \tilde{A}_{L,n}(j) + Y_{jn} \tilde{A}_{L,n}(k)] \\
&+ \left\{ \sum_{n=1}^{L-1} (-1)^{L+n} [Y_{kn} \tilde{A}_{L,n}(j) + Y_{jn} \tilde{A}_{L,n}(k)] \right\}^2
\end{aligned} \tag{54}$$

Using eq.(49) and a similar one with k replaced by j , we now write

$$\begin{aligned}
& \Delta_{1,2,\dots,L-1,k} \cdot \Delta_{1,2,\dots,L-1,j} \\
&= \sum_{n=1}^{L-1} \sum_{n'=1}^{L-1} Y_{kn} \tilde{A}_{L,n}(k) Y_{jn'} \tilde{A}_{L,n'}(j) \\
&+ Y_{kk} \cdot \Delta_{1,2,\dots,L-1} \sum_{n'=1}^{L-1} (-1)^{L+n'} Y_{jn'} \tilde{A}_{L,n'}(j) + Y_{jj} \cdot \Delta_{1,2,\dots,L-1} \sum_{n=1}^{L-1} (-1)^{L+n} Y_{kn} \tilde{A}_{L,n}(k) \\
&+ Y_{jj} Y_{kk} \cdot \Delta_{1,2,\dots,L-1}^2
\end{aligned} \tag{55}$$

We are now just one step away from proving lemma 1. We start with a determinant identity

$$\mathbf{Y} = \left(\begin{array}{c|c} A & C \\ \hline D & B \end{array} \right) = \left(\begin{array}{c|c} A & 0 \\ \hline D & I_M \end{array} \right) \left(\begin{array}{c|c} I_N & A^{-1}C \\ \hline 0 & B - DA^{-1}C \end{array} \right) \tag{56}$$

and consequently

$$\det(\mathbf{Y}) = \det(A) \cdot \det(B - DA^{-1}C). \tag{57}$$

This tells us that we can rewrite $\Delta_{1,2,\dots,L-1,k,j}$ in the following form:

$$\Delta_{1,2,\dots,L-1,k,j} = \det \left(\begin{array}{ccc|cc} & & & Y_{1k} & Y_{1j} \\ & & & \vdots & \vdots \\ (\mathbf{Y})_{L-1} & & & Y_{L-1,k} & Y_{L-1,j} \\ \hline Y_{k1} & Y_{k2} & \cdots & Y_{kk} & Y_{kj} \\ Y_{j1} & Y_{j2} & \cdots & Y_{jk} & Y_{jj} \end{array} \right) = \Delta_{1,2,\dots,L-1} \cdot \det(B - D(\mathbf{Y})_{L-1}^{-1}C) \tag{58}$$

where

$$B = \begin{pmatrix} Y_{kk} & Y_{kj} \\ Y_{jk} & Y_{jj} \end{pmatrix}, \quad D = \begin{pmatrix} Y_{k1} & Y_{k2} & \cdots & Y_{kL-1} \\ Y_{j1} & Y_{j2} & \cdots & Y_{jL-1} \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} Y_{1k} & Y_{1j} \\ Y_{2k} & Y_{2j} \\ \vdots & \vdots \\ Y_{L-1k} & Y_{L-1j} \end{pmatrix} \quad (59)$$

We see that the matrix $G \equiv D(\mathbf{Y}^{-1})_{L-1}C$ have its components as

$$\begin{aligned} G_{11} &= \sum_{n=1}^{L-1} \sum_{n'=1}^{L-1} Y_{kn} \Gamma_{n,n'} Y_{n'k} \\ G_{12} &= \sum_{n=1}^{L-1} \sum_{n'=1}^{L-1} Y_{kn} \Gamma_{n,n'} Y_{n'j} \\ G_{21} &= \sum_{n=1}^{L-1} \sum_{n'=1}^{L-1} Y_{jn} \Gamma_{n,n'} Y_{n'k} \\ G_{22} &= \sum_{n=1}^{L-1} \sum_{n'=1}^{L-1} Y_{jn} \Gamma_{n,n'} Y_{n'j} \end{aligned} \quad (60)$$

Note that $\Gamma = \mathbf{Y}^{-1}$ is the inverse of the matrix

$$\mathbf{Y} = \begin{pmatrix} Y_{11} & \cdots & Y_{1L-1} \\ \vdots & & \vdots \\ Y_{L-11} & \cdots & Y_{L-1L-1} \end{pmatrix} \quad (61)$$

One thing to remember is that

$$\Gamma_{n,n'} = (-1)^{n+n'} H_{n',n} / \Delta_{1,2,\dots,L-1} \quad (62)$$

where $H_{n',n}$ is the minor corresponding to matrix element at row n' and column n of the matrix $(\mathbf{Y})_{L-1}$.

The most important observation here is that

$$\sum_{n'=1}^{L-1} \Gamma_{n,n'} Y_{n'j} = \sum_{n'=1}^{L-1} (-1)^{n+n'} Y_{n'j} H_{n',n} / \Delta_{1,2,\dots,L-1} \quad (63)$$

Note that the RHS of the above eq is nothing but expanding the determinant of the following matrix along the n th column

$$\begin{pmatrix} Y_{11} & \cdots & Y_{1\ n-1} & Y_{1j} & Y_{1\ n+1} & \cdots & Y_{1\ L-1} \\ Y_{21} & \cdots & Y_{2\ n-1} & Y_{2j} & Y_{2\ n+1} & \cdots & Y_{2\ L-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ Y_{L-1\ 1} & \cdots & Y_{L-1\ n-1} & Y_{L-1\ j} & Y_{L-1\ n+1} & \cdots & Y_{L-1\ L-1} \end{pmatrix} \quad (64)$$

divided by $\Delta_{1,2,\dots,L-1}$. Moreover,

$$\det \begin{pmatrix} Y_{11} & \cdots & Y_{1\ n-1} & Y_{1j} & Y_{1\ n+1} & \cdots & Y_{1\ L-1} \\ Y_{21} & \cdots & Y_{2\ n-1} & Y_{2j} & Y_{2\ n+1} & \cdots & Y_{2\ L-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ Y_{L-1\ 1} & \cdots & Y_{L-1\ n-1} & Y_{L-1\ j} & Y_{L-1\ n+1} & \cdots & Y_{L-1\ L-1} \end{pmatrix} = (-1)^{L+n-1} \tilde{A}_{L,n}(j) \quad (65)$$

by the definition of $\tilde{A}_{L,n}(j)$. This is due to the fact that we need to switch columns $(L-1) - n$ times in order to move the rightmost column $L-1$ to be at column n without changing the order of the rest. And also because $(-1)^{2n} = 1$ always.

We therefore have

$$\begin{aligned} \sum_{n'=1}^{L-1} \Gamma_{n,n'} Y_{n'j} &= (-1)^{L+n-1} \tilde{A}_{L,n}(j) / \Delta_{1,2,\dots,L-1} \\ \sum_{n=1}^{L-1} Y_{jn} \Gamma_{n,n'} &= (-1)^{L+n'-1} \tilde{A}_{n',L}(j) / \Delta_{1,2,\dots,L-1} \end{aligned} \quad (66)$$

Note that the second equality came from transposing the first equality and then set the dummy variables $n' \rightarrow n$ and call n n' . Since the determinant is invariant under matrix transposition, the RHS is obtained by just swapping n and n' and also transposing the minor. Note also that $\tilde{A}_{n',L}(j)$ is the minor of the matrix element on row n' column L of the matrix \mathbf{Y}_{L-1} .

We therefore can rewrite the matrix element G in a much simpler fashion

$$\begin{aligned} G_{11} &= \sum_{n=1}^{L-1} (-1)^{L+n-1} Y_{kn} \tilde{A}_{L,n}(k) / \Delta_{1,2,\dots,L-1} = \sum_{n=1}^{L-1} (-1)^{L+n-1} Y_{nk} \tilde{A}_{n,L}(k) / \Delta_{1,2,\dots,L-1} \\ G_{12} &= \sum_{n=1}^{L-1} (-1)^{L+n-1} Y_{kn} \tilde{A}_{L,n}(j) / \Delta_{1,2,\dots,L-1} = \sum_{n=1}^{L-1} (-1)^{L+n-1} Y_{nj} \tilde{A}_{n,L}(k) / \Delta_{1,2,\dots,L-1} \\ G_{21} &= \sum_{n=1}^{L-1} (-1)^{L+n-1} Y_{jn} \tilde{A}_{L,n}(k) / \Delta_{1,2,\dots,L-1} = \sum_{n=1}^{L-1} (-1)^{L+n-1} Y_{nk} \tilde{A}_{n,L}(j) / \Delta_{1,2,\dots,L-1} \end{aligned}$$

$$G_{22} = \sum_{n=1}^{L-1} (-1)^{L+n-1} Y_{jn} \tilde{A}_{L,n}(j) / \Delta_{1,2,\dots,L-1} = \sum_{n=1}^{L-1} (-1)^{L+n-1} Y_{nj} \tilde{A}_{n,L}(j) / \Delta_{1,2,\dots,L-1} \quad (67)$$

We therefore may write $\Delta_{1,2,\dots,L-1} \cdot \Delta_{1,2,\dots,L-1,k,j}$ as

$$\begin{aligned} \Delta_{1,2,\dots,L-1} \cdot \Delta_{1,2,\dots,L-1,k,j} &= \Delta_{1,2,\dots,L-1}^2 \cdot \det(B - G) \\ &= \left[Y_{kk} \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{kn} \tilde{A}_{L,n}(k) \right] \left[Y_{jj} \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{jn} \tilde{A}_{L,n}(j) \right] \\ &\quad - \left[Y_{kj} \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{kn} \tilde{A}_{L,n}(j) \right] \left[Y_{jk} \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{jn} \tilde{A}_{L,n}(k) \right] \end{aligned} \quad (68)$$

After some algebra and using eqs(49) and (54), we can write down the expression explicitly

$$\begin{aligned} &4 \Delta_{1,2,\dots,L-1,k} \cdot \Delta_{1,2,\dots,L-1,j} - \left(\Delta_{1,2,\dots,L-1,\langle k+j \rangle} - \Delta_{1,2,\dots,L-1,k} - \Delta_{1,2,\dots,L-1,j} \right)^2 \\ &\quad - 4 \Delta_{1,2,\dots,L-1} \cdot \Delta_{1,2,\dots,L-1,k,j} \\ &= -(Y_{jk} - Y_{kj})^2 \cdot \Delta_{1,2,\dots,L-1}^2 - \left[\sum_{n=1}^{L-1} (-1)^{L+n} (Y_{kn} \tilde{A}_{L,n}(j) - Y_{jn} \tilde{A}_{L,n}(k)) \right]^2 \\ &\quad + 2Y_{jk} \cdot \Delta_{1,2,\dots,L-1} \left[\sum_{n=1}^{L-1} (-1)^{L+n} (Y_{kn} \tilde{A}_{L,n}(j) - Y_{jn} \tilde{A}_{L,n}(k)) \right] \\ &\quad - 2Y_{kj} \left[\sum_{n=1}^{L-1} (-1)^{L+n} (Y_{kn} \tilde{A}_{L,n}(j) - Y_{jn} \tilde{A}_{L,n}(k)) \right] \\ &= - \left\{ (Y_{kj} - Y_{jk}) \cdot \Delta_{1,2,\dots,L-1} + \left[\sum_{n=1}^{L-1} (-1)^{L+n} (Y_{kn} \tilde{A}_{L,n}(j) - Y_{jn} \tilde{A}_{L,n}(k)) \right] \right\}^2 \\ &= - [\det(\mathbf{Y}_{L-1}(j_{L_c}, k_{L_r})) - \det(\mathbf{Y}_{L-1}(k_{L_c}, j_{L_r}))]^2 \end{aligned} \quad (69)$$

where the matrices $\mathbf{Y}_{L-1}(j_{L_c}, k_{L_r})$ and $\mathbf{Y}_{L-1}(k_{L_c}, j_{L_r})$ look like

$$\mathbf{Y}_{L-1}(j_{L_c}, k_{L_r}) = \left(\begin{array}{ccc|c} & & & Y_{1j} \\ & & & \vdots \\ (\mathbf{Y})_{L-1} & & & Y_{L-1,j} \\ \hline Y_{k1} & Y_{k2} & \cdots & Y_{kj} \end{array} \right) \quad \mathbf{Y}_{L-1}(j_{L_c}, k_{L_r}) = \left(\begin{array}{ccc|c} & & & Y_{1k} \\ & & & \vdots \\ (\mathbf{Y})_{L-1} & & & Y_{L-1,k} \\ \hline Y_{j1} & Y_{j2} & \cdots & Y_{jk} \end{array} \right) \quad (70)$$

Note that if the matrix elements of Y is symmetric, i.e. $Y_{jk} = Y_{kj}$, the RHS of eq.(69) is identically zero. This is because the inverse of a symmetric matrix will also be symmetric. Therefore

$$\begin{aligned}
\sum_{n=1}^{L-1} (-1)^{L+n-1} Y_{kn} \tilde{A}_{L,n}(j) &= \sum_{n,n'} Y_{kn} \Gamma_{n,n'} Y_{n'j} = \sum_{n,n'} Y_{nk} \Gamma_{n',n} Y_{jn'} = \sum_{n,n'} Y_{jn'} \Gamma_{n',n} Y_{nk} \\
&= \sum_{n'=1}^{L-1} (-1)^{L+n'-1} Y_{jn'} \tilde{A}_{L,n'}(k) = \sum_{n=1}^{L-1} (-1)^{L+n'-1} Y_{jn} \tilde{A}_{L,n}(k) \quad (71)
\end{aligned}$$

We now proceed to prove lemma 2. Using the decomposition (48), and eq.(53) and its similar forms, we may write the LHS of lemma 2 as

$$\begin{aligned}
&\left[Y_{jj} \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{jn} \tilde{A}_{L,n}(j) \right] \cdot \\
&\quad \cdot \left[(Y_{kl} + Y_{lk}) \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} (Y_{kn} \tilde{A}_{L,n}(l) + Y_{ln} \tilde{A}_{L,n}(k)) \right] \\
&- \frac{1}{2} \left[(Y_{jk} + Y_{kj}) \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} (Y_{kn} \tilde{A}_{L,n}(j) + Y_{jn} \tilde{A}_{L,n}(k)) \right] \cdot \\
&\quad \cdot \left[(Y_{jl} + Y_{lj}) \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} (Y_{ln} \tilde{A}_{L,n}(j) + Y_{jn} \tilde{A}_{L,n}(l)) \right]. \quad (72)
\end{aligned}$$

Using eq.(58), we see that the corresponding G , similar to (67), of $\Delta_{1,2,\dots,L-1,j,\langle k+l \rangle}$ has the form

$$\begin{aligned}
G_{11} &= \sum_{n=1}^{L-1} (-1)^{L+n-1} Y_{jn} \tilde{A}_{L,n}(j) / \Delta_{1,2,\dots,L-1} \\
G_{12} &= \sum_{n=1}^{L-1} (-1)^{L+n-1} Y_{jn} [\tilde{A}_{L,n}(k) + \tilde{A}_{L,n}(l)] / \Delta_{1,2,\dots,L-1} \\
G_{21} &= \sum_{n=1}^{L-1} (-1)^{L+n-1} [Y_{kn} \tilde{A}_{L,n}(j) + Y_{ln} \tilde{A}_{L,n}(j)] / \Delta_{1,2,\dots,L-1} \\
G_{22} &= \sum_{n=1}^{L-1} (-1)^{L+n-1} (Y_{kn} + Y_{ln}) [\tilde{A}_{L,n}(k) + \tilde{A}_{L,n}(l)]. \quad (73)
\end{aligned}$$

Using similar expressions, we can write the RHS of lemma 2 as

$$\begin{aligned}
&\left[Y_{jj} \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{jn} \tilde{A}_{L,n}(j) \right] \cdot \\
&\quad \cdot \left[(Y_{kk} + Y_{lk} + Y_{kl} + Y_{ll}) \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (Y_{kn} + Y_{ln}) (\tilde{A}_{L,n}(k) + \tilde{A}_{L,n}(l)) \right] \\
&- \left[(Y_{jk} + Y_{jl}) \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{jn} (\tilde{A}_{L,n}(k) + \tilde{A}_{L,n}(l)) \right] \cdot \\
&\quad \cdot \left[(Y_{kj} + Y_{lj}) \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} (Y_{kn} \tilde{A}_{L,n}(j) + Y_{ln} \tilde{A}_{L,n}(j)) \right]
\end{aligned}$$

$$\begin{aligned}
& - \left\{ \left[Y_{jj} \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{jn} \tilde{A}_{L,n}(j) \right] \left[Y_{kk} \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{kn} \tilde{A}_{L,n}(k) \right] \right. \\
& \quad - \left[Y_{jk} \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{jn} \tilde{A}_{L,n}(k) \right] \left[Y_{kj} \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{kn} \tilde{A}_{L,n}(j) \right] \\
& \quad + \left[Y_{jj} \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{jn} \tilde{A}_{L,n}(j) \right] \left[Y_{ll} \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{ln} \tilde{A}_{L,n}(l) \right] \\
& \quad \left. - \left[Y_{jl} \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{jn} \tilde{A}_{L,n}(l) \right] \left[Y_{lj} \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{ln} \tilde{A}_{L,n}(j) \right] \right\} \\
& = \left[Y_{jj} \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{jn} \tilde{A}_{L,n}(j) \right] \cdot \\
& \quad \cdot \left[(Y_{kl} + Y_{lk}) \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} (Y_{kn} \tilde{A}_{L,n}(l) + Y_{ln} \tilde{A}_{L,n}(k)) \right] \\
& \quad - \left[Y_{jl} \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{jn} \tilde{A}_{L,n}(l) \right] \left[Y_{kj} \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{kn} \tilde{A}_{L,n}(j) \right] \\
& \quad - \left[Y_{jk} \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{jn} \tilde{A}_{L,n}(k) \right] \left[Y_{lj} \cdot \Delta_{1,2,\dots,L-1} + \sum_{n=1}^{L-1} (-1)^{L+n} Y_{ln} \tilde{A}_{L,n}(j) \right] \quad (74)
\end{aligned}$$

If we subtract the LHS of lemma 2 by the RHS of lemma 2, we get

$$\begin{aligned}
& LHS|_{\text{lemma 2}} - RHS|_{\text{lemma 2}} \\
& = -\frac{1}{2} \left[Y_{jk} - Y_{kj} + \sum_{n=1}^{L-1} (-1)^{L+n} (Y_{jn} \tilde{A}_{L,n}(k) - Y_{kn} \tilde{A}_{L,n}(j)) \right] \cdot \\
& \quad \cdot \left[Y_{jl} - Y_{lj} + \sum_{n=1}^{L-1} (-1)^{L+n} (Y_{jn} \tilde{A}_{L,n}(l) - Y_{ln} \tilde{A}_{L,n}(j)) \right] \\
& = -\frac{1}{2} [\det(\mathbf{Y}_{L-1}(k_{L_c}, j_{L_r})) - \det(\mathbf{Y}_{L-1}(j_{L_c}, k_{L_r}))] \cdot \\
& \quad \cdot [\det(\mathbf{Y}_{L-1}(l_{L_c}, j_{L_r})) - \det(\mathbf{Y}_{L-1}(j_{L_c}, l_{L_r}))] \quad (75)
\end{aligned}$$

Again, if the matrix \mathbf{Y} is symmetric, the difference between the LHS and RHS of lemma 2 vanishes. This finish the proof of lemma 2.

Finally, for the $L = 1$ case (where $\Delta_{1,2,\dots,L-1} = 1$ and $\Delta_{1,2,\dots,L-1,j} = Y_{jj}$), we may verify lemmas 1 and 2 by direct substitutions with the abbreviations $\Delta_{1,2,\dots,L-1} = \Delta_0$ and $\Delta_{1,2,\dots,L-1,j} = \Delta_{0,j} = \Delta_j$ when $L = 1$.

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